# Correction model Exam 03-04-2017

1. Denote $x_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $x_2 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$.

a) Define $u_1 := \frac{1}{\|x_1\|} \cdot x_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

Projection of $x_2$ onto $u_1$ :

$$p_1 = (x_2, u_1)\, u_1 = \left(\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right) \frac{1}{4} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = u_1$$

Then :

$$u_2 := \frac{1}{\|x_2 - p_1\|} \cdot (x_2 - p_1)$$

Clearly

$$x_2 - p_1 = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

so

$$\|x_2 - p_1\| = 1$$

Hence $u_2 = x_2 - p_1 = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$

Orthonormal basis for $S$ is $\left\{ \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} \right\}$

b) To be projected vector : $x = \begin{pmatrix}a\\b\\a\\b\end{pmatrix}$

Projection is

$$p = (x, u_1) \cdot u_1 + (x, u_2) \cdot u_2 =$$

$$(a+b)\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + 0 \cdot \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = \frac{a+b}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

2. $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$

Characteristic polynomial

$$p(s) = \det(A - sI) = \det\begin{pmatrix} -s & a \\ -a & -s \end{pmatrix}$$

$$= s^2 + a^2.$$

Cayley-Hamilton says $A^2 + a^2 I = 0$ so $A^2 = -a^2 I$. Also $A^3 = -a^2 A$.

We do the proof by induction on $n$.

Step 1 $n=1$ : indeed $A^2 = (-1)^1 a^2 I$ and $A^3 = (-1)^1 a^2 A$.

Step $n \to n+1$ : Assume the statements are true for $n$. Then

$$A^{2(n+1)} = A^{2n} . A^2 = (-1)^n a^{2n} I . A^2$$

$$= (-1)^n a^{2n} I . (-a^2) I = (-1)^{n+1} a^{2(n+1)} I$$

and

$$A^{2(n+1)+1} = A^{2n+1} . A^2 = (-1)^n a^{2n} A . A^2$$

$$= (-1)^n a^{2n} A^3 = (-1)^n a^{2n} (-1) a^2 A$$

$$= (-1)^{n+1} a^{2(n+1)} A.$$

3. $M = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

a) $M^T M = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$

We compute the eigenvalues of $M^T M$.
$\det(M^T M - sI) = (4-s)(13-s) - 36$
$= s^2 - 17s + 16 = (s-16)(s-1)$.
Hence $\lambda_1 = 16$ and $\lambda_2 = 1$.
The singular values are

$$\sigma_1 = \sqrt{\lambda_1} = 4 \ ; \ \sigma_2 = \sqrt{\lambda_2} = 1$$

b) We compute normalized eigenvectors of $M^T M$ corresponding to $\lambda_1$ and $\lambda_2$:

$\begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow y = 2x$.

yields $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{5}}$

$\begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow x = -2y$

yields $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{5}}$

Take $V = (v_1 \ v_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

We need to find $U = (u_1 \ u_2)$ such that $M = U \Sigma V^T \Leftrightarrow MV = U\Sigma$. This yields

$M v_1 = u_1 \sigma_1 \Rightarrow u_1 = \frac{1}{4} \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

so $u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}$

also $Mv_2 = u_2\sigma_2 \Rightarrow u_2 = \begin{pmatrix}2 & 3\\0 & 2\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}$

so $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}-1\\2\end{pmatrix}$

We take $U = \frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}$

SVD : $\begin{pmatrix}2 & 3\\0 & 2\end{pmatrix} = \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}}_{U}\underbrace{\begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}}_{\Sigma}\underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix}1 & +2\\-2 & 1\end{pmatrix}}_{V^T}$

c) Best rank 1 approximation of M is

$U\begin{pmatrix}4 & 0\\0 & 0\end{pmatrix}V^T = \frac{1}{5}\begin{pmatrix}8 & 16\\4 & 8\end{pmatrix}$

4 $A$ is normal, i.e. $A^H A = A A^H$

a) Let $U^H A U = \Lambda$, a diagonal matrix with complex diagonal entries,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad \text{with } \lambda_i \in \mathbb{C}$$

Then $A = U \Lambda U^H$ so we get

$A^H A = U \Lambda^H U^H U \Lambda U^H = U \Lambda^H \Lambda U^H$ and

$A A^H = U \Lambda U^H U \Lambda^H U^H = U \Lambda \Lambda^H U^H$.

Now

$$\Lambda^H \Lambda = \begin{pmatrix} \bar\lambda_1 & & \\ & \ddots & \\ & & \bar\lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix}$$

and

$$\Lambda \Lambda^H = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \bar\lambda_1 & & \\ & \ddots & \\ & & \bar\lambda_n \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{pmatrix}$$

We must conclude that $A^H A = A A^H$.

b) Assume $A^H A = A A^H$. We know that there exists an upper triangular matrix

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ & t_{22} & \ddots & \vdots \\ & 0 & \ddots & t_{n-1,n} \\ & & & t_{nn} \end{pmatrix}$$

and a unitary $U$ such that $A = U T U^H$

Thus we get $T = U^H A U$. We show that $T$ is normal as well:

$$T^H T = U^H A^H U U^H A U = U^H A^H A U$$
$$T T^H = U^H A U U^H A^H U = U^H A A^H U$$

Hence $T^H T = T T^H$

c) From this we can prove that, in fact, $T$ is a diagonal matrix, i.e. $t_{ij} = 0$ for $i \neq j$. Indeed from $TT^H = T^H T$ it follows

$$|t_{11}|^2 + |t_{12}|^2 + \dots + |t_{1n}|^2 = |t_{11}|^2$$
$$|t_{22}|^2 + \dots + |t_{2n}|^2 = |t_{12}|^2 + |t_{22}|^2$$
$$\vdots$$
$$|t_{nn}|^2 = |t_{1n}|^2 + |t_{2n}|^2 + \dots + |t_{nn}|^2$$

so $t_{12} = 0, \dots, t_{1n} = 0$, $t_{23} = 0, \dots, t_{2n} = 0$, etc.

5. a) Since $A$ is symmetric there exists an orthogonal matrix $U$ s.t. $U^T A U = \Lambda$ is diagonal. This implies $AU = U\Lambda$.
The diagonal elements of $\Lambda$ are the eigenvalues of $A$. Assume $\lambda$ occurs $a$ times on the diagonal, i.e. the algebraic multiplicity of $\lambda$ is $a$. The corresponding columns of $U$ are then linearly independent eigenvectors for $\lambda$, and since there are $a$ of them, the geometric multiplicity of $\lambda$ is $a$.

b) $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$

Obviously $\operatorname{rank}(A)$, which is the dimension of the span of its columns, is equal to 1

c) The dimension theorem states that

$$\dim \operatorname{Ker} A + \operatorname{rank}(A) = n$$

so $\quad \dim \operatorname{Ker} A = n-1$

d) Since $\dim \operatorname{Ker} A = n-1$ there are $n-1$ linearly independent vector $x_1, x_2 \dots x_{n-1}$ such that

$$A x_i = 0$$

equivalently

$$A x_i = 0 . x_i$$

Hence $\lambda = 0$ is an eigenvalue, and since the dimension of $\operatorname{Ker}(A - 0.I)$ is $n-1$, its geometric multiplicity is $n-1$.
By part (a) then its algebraic multiplicity is also $n-1$, since $A$ is symmetric.

e) The sum of the eigenvalues of $A$ is equal to trace $(A)$.
Since trace $(A) = n$, there is only one remaining eigenvalue : $\lambda = n$
All eigenvalues are

$$\{\underbrace{0, 0, \dots 0}_{n-1 \text{ times}}, n\}$$

6. a) Obviously the characteristic polynomial of $M$ is $(1-s)^2(2-s)$. Hence the eigenvalues are $\lambda=1$ with algebraic multiplicity 2 and $\lambda=2$ with algebraic multiplicity 1.

b) The geometric multiplicity of $\lambda=1$ equals the dimension of $\ker(M-I)$ and likewise of $\lambda=2$ the dimension of $\ker(M-2I)$. We compute these kernels for different choices of $a, b$:

$\underline{\lambda=1}$ $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \ker(M-I) \iff \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{array}{l} a x_2 = 0 \\ b x_3 = 0 \\ 2 x_3 = 0 \end{array}$$

Two cases: $a=0$ : this yields $x_3=0$
$x_1, x_2$ arbitrary
$a \neq 0$ : this yields $x_3=0, x_2=0$
$x_1$ arbitrary

So: if $a=0$ then $\lambda=1$ has geometric multiplicity 2, if $a\neq 0$ geometric multiplicity is 1.

$\lambda=2$ $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \ker(M-2I) \iff \begin{pmatrix} -1 & a & 0 \\ 0 & -1 & b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow \quad -x_1 + a x_2 = 0$
$\quad\quad -x_2 + b x_3 = 0$

Two cases: $b = 0$: this yields $x_2 = 0$
$x_1 = 0$, $x_3$ arbitrary

$b \neq 0$: this yields $x_2$ arbitrary, $x_1 = a x_2$ and $x_3 = \frac{1}{b} x_2$.

So: In both cases, $b \neq 0$ and $b = 0$, the geometric multiplicity of $\lambda = 2$ equals $1$.

$\boxed{\text{see further for alternative solution}}$

c) From the above we can deduce the Jordan form for the different $a, b$:
For each $b \in \mathbb{R}$ we have

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{if} \quad a = 0$$

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{if} \quad a \neq 0.$$

Alternative solution to b):

$$M = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 2 \end{pmatrix}$$

We want to compute the geometric multiplicities of $\lambda = 1$ and $\lambda = 2$

$\lambda = 1$ $\quad M - I = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 2 \end{pmatrix}$

if $a = 0$, $\text{rank}(M - I) = 1$ so $\dim \ker(M - I) = 2$

if $a \neq 0$, $\text{rank}(M - I) = 2$ so $\dim \ker(M - I) = 1$

$\lambda = 2$ $\quad M - 2I = \begin{pmatrix} -1 & a & 0 \\ 0 & -1 & b \\ 0 & 0 & 0 \end{pmatrix}$

For all $a$ and $b$, $\text{rank}(M - 2I) = 2$ so $\dim \ker M - 2I = 1$

Conclusion: $\lambda = 1$ : $a = 0$ geo. mult is 2
$a \neq 0$ geo mult is 1
$\lambda = 2$ : geometric mult is 1 for all $a$ and $b$.

By the way: for $\lambda = 2$ the geometric multiplicity is of course also 1 because $\lambda = 2$ occurs only <u>once</u>